Bachelor of Commerce

Semester – I

Paper Code –

BUSINESS MATHEMATICS-I

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PAPER CODE:

Theory Paper Max Marks: 80
Time: 3Hrs **Internal marks: 20**

Unit I

Calculus: (Problems and theorems involving trigonometrically ratios are not to be done). **Differentiation:** Partial derivatives up to second order; Homogeneity of functions and Euler's theorem; total differentials, Differentiation of implicit function with the help of total differentials. Maxima and Minima; Cases of one variable involving second or higher order derivatives; Cases of two variables involving not more than one constraint.

Unit-II

Integration: Integration as anti-derivative process; Standard forms; Methods of integration-by substitution, by parts, and by use of partial fractions; Definite integration; Finding areas in simple cases; Consumers and producers surplus; Nature of Commodities learning Curve; Leontiff Input-Output Model.

Unit-III

Matrices: Definition of matrix; Types of matrices; Algebra of matrices;

Unit-IV

Determinants: Properties of determinants; calculation of values of determinants up to third order; Adjoint of a matrix, through Adjoint and elementary row or column operations; Solution of system of linear equations having unique solution and involving not more than three variables.

Suggested Readings:

1. *Allen B.G.D: Basic Mathematics; Mcmillan, New Delhi.*
2. *Volra. N. D. Quantitative Techniques in Management, Tata McGraw Hill, New Delhi. Kapoor V.K. Business*
3. *Mathematics: Sultan chand and sons, Delhi.*

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1

Calculus and Differentiation

Structure

1.1. Introduction.

1.2. Differentiation.

1.3. Differentiation of Logarithmic and Exponential functions.

1.4. Partial derivatives.

1.5. Total Differentials.

1.6. Implicit Functions.

1.7. Homogeneous Functions.

1.8. Local Maxima and Local Minima.

1.9. Check Your Progress.

1.10. Summary.

1.1. Introduction. This chapter contains many important results related derivatives, partial derivatives and their use to obtain extreme values of a function.

1.1.1. Objective. The objective of these contents is to provide some important results to the reader like:

(i) Derivatives.

(ii) Partial Derivatives.

(iii) Euler's Theorem.

(iv) Maxima and Minima

1.1.2. Keywords. Continuity, Differentiation, Partial Differentiation, Homogeneous Functions.

1.2. Differentiation.

Differentiation is the technique of determining the derivatives of continuous functions and derivative is the limit of average rate of change in the dependent function following a change in the value of the variable. Very small change in the value of independent variable is accompanied by a very small change in the value of dependent variable.

Mathematically, we say that y is a function of x or y = f(x). The set of all permissible values of x is called **Domain** of the function and the set of corresponding values of y is called the **Range** of the function.

1.2.1. Derivative of a function.

To obtain the derivative of a given function:

Let

$$y = f(x) \tag{1}$$

be the given function of *x*.

Given a small increment δx in *x,* assume δy be the corresponding increment in *y* so that

$$y + \delta y = f(x + \delta x) \tag{2}$$

Subtract (1) from (2), we get

$$\delta y = f(x+\delta x) - f(x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Proceeding to limits $\delta x \to 0$ which gives

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

On evaluating the limit of equation (5), we get the value of $\frac{dy}{dx}$

This method of obtaining derivative is known as **differentiation from first principle** or by **ab-initio method** or **from delta method** or **from definition**.

1.2.2. Example. The derivative of x^n is nx^{n-1} where n is fixed number, integer or rational.

Solution. Let

$$y = x^n \tag{1}$$

Let δx be a small increment in *x* and δy be the corresponding increment in *y,* then

$$y + \delta y = (x + \delta x)^n \tag{2}$$

Subtracting (1) from (2), we get

$$\delta y = (x+\delta x)^n - x^n$$

dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}$$

Proceeding to limits as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = n.x^{n-1}$$

Using the fact that $\lim_{x \to a} \frac{x^n - a^n}{x-a} = n.a^{n-1}$

Hence $\frac{d}{dx}(x^n) = n.x^{n-1}$

1.2.3. Example. Find the derivatives of the functions

(i) x^{10} (ii) x^{-9} (iii) $x^{2/3}$

Solution.

(i) Let $y = x^{10}$, then $\frac{dy}{dx} = 10.x^9$.

(ii) Let $y = x^{-9}$, then $\frac{dy}{dx} = -9.x^{-10}$.

(iii) Let $y = x^{2/3}$, then $\frac{dy}{dx} = \frac{2}{3}.x^{-\frac{1}{3}}$.

1.2.4. Example. The derivative of $(ax+b)^n$ is $na(ax+b)^{n-1}$

Solution. Let $y = (ax+b)^n$ (1)

Let δx be a small increment in *x* and δy be the corresponding increment in *y,* then

$$y + \delta y = [a(x+\delta x)+b]^n = [(ax+b)+a\delta x]^n \qquad (2)$$

Subtracting (1) from (2), we get

$$\delta y = [(ax+b)+a\delta x]^n - (ax+b)^n$$

Dividing both sides by δx, we have

$$\frac{\delta y}{\delta x} = \frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{\delta x} = a\frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{a\delta x}$$

Proceeding to limits as $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} a\frac{[(ax+b)+a\delta x]^n - (ax+b)^n}{a\delta x} = na(ax+b)^{n-1}$$

Hence, $\frac{dy}{dx} = na(ax+b)^{n-1}$.

1.2.5. Example. Find the derivative of $x^2 + 3x$ w.r.t. 'x' by using the first principle.

Solution. Let $y = x^2 + 3x$ (1)

Let δx be small increment in x and δy be the corresponding increment in y, then

$$y + \delta y = (x + \delta x)^2 + 3(x + \delta x) \qquad (2)$$

Subtracting (1) from (2), we get

$$\delta y = (\delta x)^2 + 2x\delta x + 3(\delta x)$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \delta x + 2x + 3$$

Proceeding to limits as $\delta x \to 0$, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 2x + 3$$

Hence $\dfrac{dy}{dx} = 2x + 3$

1.2.6. Example. Find the derivative of $x + \dfrac{1}{x}$ w.r.t. 'x'.

Solution. Let $y = x + \dfrac{1}{x} = x + x^{-1}$ then, $\dfrac{dy}{dx} = 1.x^{1-1} + (-1)x^{-1-1} = 1 - \dfrac{1}{x^2}$.

1.2.7. Example. Differentiate $\sqrt{x} + \dfrac{1}{x^{\frac{3}{2}}}$ w.r.t. 'x.

Solution. Let $y = \sqrt{x} + \dfrac{1}{x^{\frac{3}{2}}} = x^{\frac{1}{2}} + x^{-\frac{3}{2}}$, then $\dfrac{dy}{dx} = \dfrac{1}{2}x^{\frac{1}{2}-1} + \left(-\dfrac{3}{2}\right)x^{-\frac{3}{2}-1} = \dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{3}{2}x^{-\frac{5}{2}}$.

1.2.8. Results.

1. $\dfrac{d}{dx}(c) = 0$ where c is constant function.
2. $\dfrac{d}{dx}[c.\, f(x)] = c\,.\dfrac{d}{dx}[f(x)]$ where 'c' is constant.
3. If u and v are differentiable functions of 'x' then $\dfrac{d}{dx}(u+v) = \dfrac{d}{dx}(u) + \dfrac{d}{dx}(v)$ and $\dfrac{d}{dx}(u-v) = \dfrac{d}{dx}(u) - \dfrac{d}{dx}(v)$.
4. **Product Rule for differentiation.** If u,v and w are functions of x then

   (i) $\dfrac{d}{dx}(u.v) = u\dfrac{d}{dx}v + v\dfrac{d}{dx}u$

(ii) $\frac{d}{dx}(u.v.w) = vw\frac{d}{dx}(u) + wu\frac{d}{dx}(v) + uv\frac{d}{dx}(w)$

5. **Quotient Rule for Differentiation.** If u and v are functions of x and $v \neq 0$ then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}v}{v^2}.$$

6. **Chain Rule.** If $y = f(u)$ and $u = \phi(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

7. If $y = f(u)$, $u = g(v)$ and $v = \phi(x)$ are three differentiable function, then by chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left[\frac{du}{dv}\frac{dv}{dx}\right] = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

that is, $$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dx}.$$

1.2.9. Example. Differentiate $(x+a)^m(x+b)^n$ w.r.t. x

Solution. Let $y = (x+a)^m\ (x+b)^n$, then using the product rule of differentiation, we have

$$\frac{dy}{dx} = (x+a)^m\frac{d}{dx}(x+b)^n + (x+b)^n\frac{d}{dx}(x+a)^m$$

$$= (x+a)^m.n(x+b)^{n-1}.1 + (x+b)^n.m(x+b)^{m-1}.1$$

$$= (x+a)^{m-1}.(x+b)^{n-1}\left[n(x+a) + m(x+b)\right]$$

$$= (x+a)^{m-1}(x+b)^{n-1}\left[(m+n)x + an + bm\right].$$

1.2.10. Example. Differentiate $\frac{ax+b}{cx+d}$ w.r.t. x

Solution. Let $y = \frac{ax+b}{cx+d}$, then by quotient rule of differentiation, we have

$$\frac{dy}{dx} = \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2}$$

$$= \frac{(cx+d)(a+0) - (ax+b)(c+0)}{(cx+d)^2} = \frac{acx + ad - acx - bc}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}.$$

1.2.11. Example. If $y = u^2 + u + 6$, $u = v^2 + 75$, $v = 6x + 17$, then find $\frac{dy}{dx}$.

Solution. We have $\frac{dy}{du} = 2u + 1$, $\frac{du}{dv} = 2v$, $\frac{dv}{dx} = 6$

By chain rule $\dfrac{dy}{dx}=\dfrac{dy}{du}\,\dfrac{du}{dv}\,\dfrac{dv}{dx}=(2u+1)\,(2v)(6)=\left[2\left(v^2+75\right)+1\right]12\,v$

$$=12\,v\left(2v^2+151\right)=12\left(6x+17\right)\left[2\left(6x+17\right)^2+11\right].$$

1.2.12. Exercise. Find the derivatives of the following functions w.r.t. x by using first principle.

1.	x^4	**2.**	$9x+10$	**3.**	$x^2+10x+80$
4.	$x^{\frac{4}{5}}$	**5.**	$x^{-\frac{1}{4}}$	**6.**	$\dfrac{2}{\sqrt{x}}$
7.	$\dfrac{x+1}{x^2}$	**8.**	$x^{\frac{1}{4}}+\dfrac{1}{x}$	**9.**	$\sqrt{2x+7}$
10.	$(7x+6)^{-4}$	**11.**	$\dfrac{ax+b}{cx+d}$	**12.**	$2x+\dfrac{1}{2x^{\frac{3}{2}}}$.

13. If $y=1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^n}{n!}$, then show that $\dfrac{dy}{dx}-y+\dfrac{x^n}{n!}=0$.

Answers.

1.	$4x^3$	**2.**	9	**3.**	$2x+10$
4.	$\dfrac{4}{5x^{\frac{1}{5}}}$	**5.**	$-\dfrac{1}{4}x^{-\frac{5}{4}}$	**6.**	$-\dfrac{1}{x^{\frac{3}{2}}}$
7.	$-\dfrac{1}{x^2}-\dfrac{1}{x^3}$	**8.**	$\dfrac{1}{4x^{\frac{3}{4}}}-\dfrac{1}{x^2}$	**9.**	$\dfrac{1}{\sqrt{2x+7}}$
10.	$-\dfrac{28}{\left(7x+6\right)^5}$	**11.**	$\dfrac{ad-bc}{(cx+d)^2}$	**12.**	$2-\dfrac{1}{2x^{\frac{5}{2}}}$.

1.2.13. Exercise. Differentiate w.r.t. x the following:

1.	$(x^2+1)(x^2+x+4)$	**2.**	$x(x-3)(x^2+x)$	**3.**	$\dfrac{x+3}{x^2+1}$
4.	$\dfrac{3x+2}{(x+5)(2x+1)+3}$	**5.**	$y=v^3+2v^2+5$, $v=3u+1$ and $u=9x+1$		

Answers.

1.	$4x^3+3x^2+10x+1$	**2.**	$2x(2x^2-3x-3)$	3.	$\dfrac{1-6x-x^2}{(x^2+1)^2}$
4.	$\dfrac{-6x^2-8x+2}{(2x^2+11x+8)^2}$	5.	$27\left(2187\,x^2+756\,x+64\right)$		

1.3. Differentiation of Logarithmic and Exponential functions.

1.3.1. Exponential function. If 'a' be any positive real number then $y = a^x$ is called an exponential function where $x \in R$. When $a = e$, then $y = e^x$ is called exponential function.

1.3.2. Derivative of Exponential function. $\dfrac{d}{dv}(a)^u = a^u \log_e a \dfrac{du}{dv}$.

Also when $a = e$, then $\dfrac{d}{dv}a^x = a^x \log_e a$.

In particular, when $u = x$ then $\dfrac{d}{dx}(e^x) = e^x$.

1.3.3. Derivative of Logarithmic Function.

If u is any differentiable function of x, then

$$\frac{d}{dx}\log_a u = \frac{1}{u}\log_a e \frac{d}{dx}(u)$$

In particular, when $u = x$ then $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}$.

1.3.4. Some Properties of Logrithm.

(i) $\log_a(m \cdot n) = \log_a m + \log_a n$

(ii) $\log_a \dfrac{m}{n} = \log_a m - \log_a n$

(iii) $\log_a m^n = n \log_a m$

(iv) $\log_a m = \log_b m \cdot \log_a b$

(v) $\log_b a = \dfrac{\log a}{\log b}$

1.3.5. Example. Differentiate the following functions w.r.t. 'x'

(i) e^{5x+3} **(ii)** e^{e^x} **(iii)** 8^{5x+7}

Solution. (i) Let $y = e^{5x+3}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{5x+3} = e^{5x+3}\dfrac{d(5x+3)}{dx} = 5e^{5x+3}$.

(ii) Let $y = e^{e^x}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}e^{e^x} = e^{e^x}\dfrac{de^x}{dx} = e^x e^{e^x}$.

(iii) Let $y = 8^{5x+7}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(8^{5x+7}) = 8^{5x+7}\dfrac{d(5x+7)}{dx}\log 8 = (5\log 8)8^{5x+7}$.

1.3.6. Exercise. Differentiate the following functions w.r.t. 'x':

1. $\log\left(x + \sqrt{a^2 + x^2}\right)$ **2.** $\log\left[\log\left(\log x^4\right)\right]$ **3.** $e^x \log\left(1 + x^2\right)$

4. **(i)** If $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, then prove that $\dfrac{dy}{dx}=1-y^2$

(ii) If $y=\log\left(\sqrt{x-1}-\sqrt{x+1}\right)$, then prove that $\dfrac{dy}{dx}=\dfrac{-1}{2\sqrt{x^2-1}}$

(iii) If $y=(x-1)\log(x-1)-(x+1)\log(x+1)$, then prove that $\dfrac{dy}{dx}=\log\left(\dfrac{x-1}{x+1}\right)$.

Answer.

1. $\dfrac{1}{\sqrt{a^2+x^2}}$ 2. $\dfrac{4}{x\left(\log x^4\right)\left[\log\left(\log x^4\right)\right]}$ 3. $e^x\left[\dfrac{2x}{1+x^2}+\log\left(1+x^2\right)\right]$

1.4. Partial derivatives.

Let f be a function of two or more variables. The derivative of f w.r.t. one independent variable, while considering all other independent variables constant, is called the partial derivative of f w.r.t. that variable.

If $f(x,y)$ is a function of two independent variables x and y, then the partial derivative of $f(x,y)$ w.r.t. x is the derivative of $f(x,y)$ when y is regarded as constant. It is denoted by $\dfrac{\partial f}{\partial x}$ or f_x or $D_x f$. Thus,

$$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}.$$

Similarly, the partial derivative of $f(x,y)$ w.r.t. y is defined as $\dfrac{\partial f}{\partial y}=\lim\limits_{k\to 0}\dfrac{f(x,y+k)-f(x,y)}{k}$.

This definition can be extended to a function of having more than two independent variables.

1.4.1. Second order partial derivatives. If $f(x,y)$ has partial derivatives at each point, then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are themselves functions of x and y, which may also have partial derivatives, known as second order partial derivatives. These second derivatives are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$

1.4.2. Remark. Generally $\frac{\partial^2 f}{\partial y\,\partial x} \neq \frac{\partial^2 f}{\partial x\,\partial y}$. But here we will deal with only those functions for which $\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}$. Also, all the results stated for ordinary differentiation, like chain rule, product rule, quotient rule etc., are valid for partial differentiation.

1.4.3. Example. Find the all the first and second order partial derivatives of $x^5 + y^5 - 2ax^2y + xy$**.**

Solution. Let $u = x^5 + y^5 - 2ax^2y + xy$, then

$$\frac{\partial u}{\partial x} = 5x^4 - 4axy + y, \qquad \frac{\partial u}{\partial y} = 5y^4 - 2ax^2 + x$$

$$\frac{\partial^2 u}{\partial x^2} = 20x^3 - 4ay, \qquad \frac{\partial^2 u}{\partial y^2} = 20y^3$$

$$\frac{\partial^2 u}{\partial x\partial y} = -4ax + 1, \qquad \frac{\partial^2 u}{\partial y\partial x} = -4ax + 1$$

1.4.4. Exercise.

1. If $u = \log\left(x^2 + y^2\right)$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

2. For the function $z = x^y + y^x$ verify that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.
3. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$, prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$.
4. If $u = 3\left(lx + my + nz\right)^2 - \left(x^2 + y^2 + z^2\right)$ and $l^2 + m^2 + n^2 = 1$. Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

5. Find the first and second order partial derivative of $y\log x$.
6. If $z^3 - xz - y = 0$, prove that $\frac{\partial^2 z}{\partial x\,\partial y} = -\frac{3z^2 + x}{\left(3z^2 - x\right)^3}$.
7. Find the value of $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$ when $a^2x^2 + b^2y^2 - c^2z^2 = 0$.
8. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x\,\partial y\,\partial z} = \left(1 + 3xyz + x^2y^2z^2\right)e^{xyz}$.If $u = f\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Now $f'(x) = 0 \quad \Rightarrow \quad 3x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{1}{3}, 1$

Since $\frac{1}{3}, -1 \in \left[-2, \frac{5}{2}\right]$, therefore, $x = \frac{1}{3}$ and $x = -1$ are the only stationary points. So we need to find the value of $f(x)$ at $x = -2, -1, \frac{1}{3}, \frac{5}{2}$.

Here, $f(-2) = -\frac{7}{4}$, $f(-1) = \frac{5}{4}$, $f\left(\frac{1}{3}\right) = \frac{7}{108}$, $f\left(\frac{5}{2}\right) = \frac{157}{8}$.

Hence, absolute maximum value of $f(x)$ is $\frac{157}{8}$ at $x = \frac{5}{2}$ and absolute minimum value is $-\frac{7}{4}$ at $x = -2$.

1.8.13. Remarks.

1. Area and parameter of a rectangle of sides x and y are xy and $2(x + y)$.
2. Area and parameter of a square of side x are x^2 and $4x$.
3. Area and circumference of a circle of radius r are πr^2 and $2\pi r$.
4. Volume and Surface area of a cube of edge length x are x^3 and $6x^2$.
5. Volume and Surface area of a cuboid of edges of length x, y and z are xyz and $2(xy + yz + zx)$.
6. Volume and Surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$.
7. Volume, Surface area and Curved Surface area of a right circular cylinder of base radius r and height h are $\pi r^2 h$, $2\pi rh + 2\pi r^2$ are $2\pi rh$ respectively.
8. Volume, Surface area and Curved Surface area of a right circular cone of height h, slant height l and radius of base r are $\frac{1}{3}\pi r^2 h$, $\pi rl + \pi r^2$ and πrl respectively.

1.8.14. Example. Divide 30 into two parts such that their product is maximum .

Solution. Let one part is x. Then, second part will be $30 - x$. Let the product of two parts be P. Then,

$$P = x(30 - x)$$

$$\Rightarrow \quad P = 30x - x^2 \quad \Rightarrow \quad \frac{dP}{dx} = 30 - 2x$$

For stationary points, we take

$$\frac{dP}{dx} = 0 \quad \Rightarrow \quad 30 - 2x = 0 \quad \Rightarrow \quad x = 15$$

Now, $\frac{d^2P}{dx^2} = -2 < 0$ when $x = 15$. Therefore, P is maximum when $x = 15$, that is, P is maximum when first part is 15 then second part is also 15.

1.8.15. Example. Find two positive numbers with sum 35 and product of square of one and fifth power of second is maximum.

Solution. Assume the numbers are x and y. Then, we have,

$$x+y=35$$

Let $P=x^2y^5$ or $P=x^2y^5=(35-y)^2y^5$

Then, $\frac{dP}{dy}=(35-y)(y^4)\left[175-7y\right]=7(35-y)y^4(25-y)$

For stationary points, we have

$$\frac{dP}{dy}=0 \quad \Rightarrow \quad 7(35-y)\ y^4\ (25-y)=0 \quad \Rightarrow \quad y=0, 25, 35$$

But $y = 0$ and $y = 35$ are not possible. So $y = 25$. Also,

$$\frac{d^2P}{dy^2} = -7y^4(25-y) + 28(35-y)y^3(25-y)-7(35-y)y^4.$$

At $y=25$, $\frac{d^2P}{dy^2}=-70(25)^4<0$.

Thus, P is maximum when $y = 25$ and so $x = 35 - 25 = 10$.

1.8.16. Example. Show that all the rectangles with a given perimeter, the square has the largest area.
Solution. Let x and y be the lengths of two sides of a rectangle of fixed perimeter P and let A be its area. Then, we have $P = 2(x+y)$ and $A = xy$.

Now, $P = 2(x+y) \Rightarrow y=\frac{P}{2}-x$. Then, $A = xy = x\left(\frac{P}{2}-x\right)=\frac{P}{2}x-x^2$, and so

$$\frac{dA}{dx}=\frac{P}{2}-2x \quad \text{and} \quad \frac{d^2A}{dx^2}=-2$$

For stationary point, take $\frac{dA}{dx}=0 \Rightarrow x=\frac{P}{4}$

For $x=\frac{P}{4}$, $\frac{d^2A}{dx^2}=-2<0$. Therefore, A is maximum when $x=\frac{P}{4}$. Also, we have $y=\frac{P}{4}$.

Hence A is maximum when $x=y=\frac{P}{4}$, that is, when rectangle is a square.

1.8.17. Exercise. Determine the local maximum and local minimum values, if any, for the following functions:

(i) $f(x)=x^3-6x^2+9x-8$ **(ii)** $f(x)=\frac{x^4}{x-1}$, $x\neq1$

(iii) $f(x)=3x^4-2x^3-6x^2+6x+1$ **(iv)** $f(x)=(x-1)^3(x+1)^2$

(v) $f(x)=x+\frac{1}{x}$ **(vi)** $f(x)=\frac{4}{x+2}+x$

(vii) $f(x)=(x-3)^4$

Answers. (i) $x = 1$ is a point of local maxima and local maximum value is -4, and $x = 3$ is a point of local minima and local minimum value is -8.

(ii) $x = 0$ is the point of local maxima and local maximum value is 0, and $x = \frac{4}{3}$ is the point of local minima and local minimum value is $\frac{256}{27}$.

(iii) $x = \frac{1}{2}$ is the point of local maxima and local maximum value is $\frac{39}{16}$, and $x = 1, -1$ is the point of local minima and local minimum value is 2, -6 respectively.

1.8.18. Exercise. Prove that following functions do not have maxima or minima:

(i) $f(x) = e^{ax+b}$ **(ii)** $f(x) = \log(2x+5)$

1.8.19. Exercise.

1. Find the absolute maximum value and the absolute minimum value of the following functions:

(i) $f(x) = x - x^2$ in $[-2, 5]$ **(ii)** $f(x) = (x-2)\sqrt{x-1}$ in $[1, 10]$

(iii) $f(x) = x^3 - 12x + 551$ in $[-3, -1]$ **(iv)** $f(x) = x^3 - 12x^2 + 18$ in $[1, 10]$

1.8.20. Exercise.

1. Among all pairs of positive numbers with product 256, find those having minimum sum.
2. Find two positive numbers with sum 16 and the sum of whose squares in minimum.
3. Show that of all the rectangles of given area, the square has the smallest perimeter.
4. Show that of all the rectangles inscribed in a given circle, the square has the maximum area.
5. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. Find the side of the square to be cut off so that the volume of the box is maximum possible.
6. Show that for a cone of given volume, curved surface area will be minimum when the height is $\sqrt{2}$ times the radius of the base.
7. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'a' is $\frac{2a}{\sqrt{3}}$.
8. A box with a square top and bottom is to be made to contain 500cc. Material for top and bottom costs Rs. 4 per sq. unit and the material for sides costs Rs. 1 per sq. cm. What is the cost of the least expensive box that can be made?
9. Find two numbers whose sum is 24 and whose product is maximum.
10. Find two numbers x and y such that $x + y = 60$ and xy^3 is maximum.
11. Prove that the area of a right angled triangle of given hypotenuse in maximum when the triangle is isosceles.

12. Show that a cylinder of a given volume which is open at the top, has minimum total surface area if its height is equal to the radius of its base.

13. Show that the height of a cylinder, which is open at the top, having a given surface and maximum volume, is equal to the radius of the base.

14. The cost C of manufacturing an article is given by the formula $C = 3x^2 + \dfrac{48}{x} + 5$ where x is the number of articles manufactured. Find the minimum value of C.

15. Find the maximum profit that a company can make, if the profit function is given by $P(x) = 41 - 24x - 18x^2$.

Answers.

1.	Both parts 16.	**2.**	Both 8.	**5.**	The side of the squares is 5 cm.
8.	Rs. 600	**9.**	12, 12	**10.**	x = 15, y = 45
16.	49				

1.8.21. Finding Maxima and Minima in cases of two variables involving not more than one constraint.

If $f(x,y)$ be a function of two independent variables x and y. Then $f(x,y)$ is said to have maximum or minimum value at the point (a, b) if $f(a,b) > f(a+h,b+k)$ or $f(a,b) < (a+h,b+k)$ for small values of h and k, positive or negative.

1.8.22. Remark. Maximum or minimum value of a function $f(x,y)$ is called its extreme value. For an extreme value at (a,b), the difference $f(a+h,b+k) - f(a,b)$ must have the same sign for all values of h & k

1.823. Necessary conditions for the function $f(x,y)$ to have an extreme value at (a,b).

Due to Taylor's theorem for function of two variables

$$f(a+h,b+k) - f(a,b) = h\frac{\partial f}{\partial x}(a,b) + k\frac{\partial f}{\partial y}(a,b)$$

$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b) + 2hk\,\frac{\partial^2 f}{\partial x\,\partial y}(a,b) + k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right] + \ldots$$

Now h and k are small enough, so second and higher degree terms of h and k may be neglected. Thus the sign of $f(a+h,\, b+k) - f(a,b)$ will be similar to that of $h\dfrac{\partial f}{\partial x}(a,b) + k\dfrac{\partial f}{\partial y}(a,b)$. For having an extreme value, $f(a+h\ ,\ b+k) - f(a,b)$ must have the same sign for all small values of h and k, positive or negative. This is possible only when $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)=0$. Thus the necessary conditions for $f(x,y)$ to have extreme value at (a,b) are

$$\frac{\partial f}{\partial x}(a,b)=0=\frac{\partial f}{\partial y}(a,b)$$

1.8.24. Stationary point. The points satisfying the condition $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ are called stationary point of the function $f(x,y)$.

Saddle point. The stationary point of function $f(x,y)$, is the point obtained from $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ at which the function has neither maximum value nor minimum value is called saddle point of $f(x,y)$.

1.8.25 Condition for a function (*x*, *y*) to have maximum or minimum at a point.

The Taylor's theorem for function of two variables is given by

$$f(a+h\ ,\ b+k\ )-f(a,b)=h\frac{\partial f}{\partial x}(a,\ b)+k\ \frac{\partial f}{\partial y}(a,b)$$
$$+\frac{1}{2!}\left[\ h^2\frac{\partial^2 f}{\partial x^2}(a,b)\ +\ 2hk\frac{\partial^2 f}{\partial x\partial y}(a,\ b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\ \right]+\ldots$$

For $f(x,y)$ to have extreme value at (*a*, *b*), we must have $\frac{\partial f}{\partial x}(a,b)=0$ and $\frac{\partial f}{\partial y}(a,b)\ =0$. Using these

$$f(a+h\ ,\ b+k)-f(a,b)=\frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2}(a,b)+2hk\frac{\partial^2 f}{\partial x\partial y}(a,b)+k^2\frac{\partial^2 f}{\partial y^2}(a,b)\right]+\ldots$$

Let us take $A=\frac{\partial^2 f}{\partial x^2}(a,b), B=\frac{\partial^2 f}{\partial x\partial y}(a,b), C=\frac{\partial^2 f}{\partial y^2}(a,b)$. Then

$$f(a+h\ ,\ b+k)-f(a,b)=\frac{1}{2!}\left(Ah^2+hkB+Ck^2\right)+\ldots$$
$$=\frac{1}{2!A}\left[A^2h^2+2hk\ A\ B+ACk^2\right]+\ldots$$
$$=\frac{1}{2!A}\left[A^2h^2\ +\ 2hk\ A\ B\ +\ B^2k^2\ +\ ACk^2\ -\ B^2k^2\right]+\ldots$$
$$=\frac{1}{2!A}\left[(Ah\ +\ Bk)^2\ +\ \left(AC-B^2\right)k^2\right]+\ldots$$

Neglecting higher degree terms, the sign of $f(a+h,b+k)-f(a,b)$ depends on $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$. In this, $(Ah+Bk)^2$ and k^2 are positive for all *h* and *k*. The sign of $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$ depends on the signs of $AC-B^2$ and A.

The following cases are to be considered:

Case I. If $(AC-B^2)>0$, then the square bracket in the expression $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$ is positive and the sign depends an *A* only. When *A* > 0, then the expression $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$ is positive and hence $f(a+h,\,b+k)-f(a,b)>0$, which implies $f(x,y)$ has minimum value at (*a*, *b*). When *A* < 0, then the expression $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$ is negative and hence $f(a+h,\,b+k)-f(a,b)<0$, which implies $f(x,y)$ has maximum value at (a,b).

Case II. If $(AC-B^2)<0$, then the sign of expression $\frac{1}{2!A}\left[(Ah+Bk)^2+(AC-B^2)k^2\right]$ depends on small values of *h* and *k* and can have different signs for different values of *h* and *k*. Hence $f(x,y)$ has neither maximum nor minimum at (a,b). Such a point (*a*, *b*) is called saddle point.

Case III. If $AC-B^2=0$, then $f(a+h,\,b+k)-f(a,b)=\frac{1}{2!A}(Ah+Bk)^2$, which may vanish for values of (*h*, *k*) for which $Ah+Bk=0$. Then sign of $f(a+h,\,b+k)-f(a,b)$ will depend upon the next term of Taylor's expansion. This is the doubtful case and requires further investigation.

1.8.26. Conclusions. Concluding the above theorem the following procedure is used to obtain maximum and minimum of a function $f(x,y)$

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and then solve the equations $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. Assume the points obtained are $(x_1,y_1),(x_2,y_2),\ldots$.

2. Then, find $\frac{\partial^2 f}{\partial x^2}=A,\ \frac{\partial^2 f}{\partial x\partial y}=B,\ \frac{\partial^2 f}{\partial y^2}=C$ and calculate the values of A, B, C at the point $(x_1,y_1),(x_2,y_2),\ldots$.

3. For (x_i,y_i),

(i) If $AC-B^2>0$ and $A<0$, then $f(x,y)$ has miaximum value at (x_1,y_1)

(ii) If $AC-B^2>0$ and $A>0$, then $f(x,y)$ has minimum value at $(x_1\ ,\ y_1)$

(iii) If $AC-B^2<0$, then $(x\ ,\ y)$ has neither maximum value nor minimum value at $(x_1\ ,\ y_1)$ and $(x_1\ ,\ y_1)$ is a saddle point

(iv) If $AC-B^2=0$, then the case is doubtful and we check the sign of $f(a+h\ ,b+k)-f(a,b)$ for small values of h and k . $f(x,y)$has a maximum or minima according as $f(a+h,\ b+k)-f(a,b)$ is < 0 or > 0.

1.8.27. Example. Obtain extreme values for the function

$$f(x,y)=x^3+y^3-63(x+y)+12xy\,\textbf{.}$$

Solution. Let $f(x,y)=x^3+y^3-63(x+y)+12x\,y$.

Then, $\dfrac{\partial f}{\partial x}=3x^2-63+12y=3(x^2+4y-21)$ and $\dfrac{\partial f}{\partial x}=3x^2-63+12y=3(y^2+4x-21)$.

Then, $\dfrac{\partial f}{\partial x}=0$ and $\dfrac{\partial f}{\partial y}=0$ implies

$$x^2+4y-21=0$$

$$y^2+4x-21=0$$

Solving these, we get $(x-y)[x+y-4]=0$, that is, $x-y=0$ or $x+y-4=0$

When $x+y=4$, then we have x = 5, -1. Now for $x=5$, $y=-1$ and $x=-1$, $y=5$.

Thus points are $(5,-1)$ and $(-1,\ 5)$.

When $x-y=0$, that is, $x=y$. We have x = -7, 3. Now for $y=-7$, $x=-7$ and $y=3$, $x=3$.

Thus points are $(-7,-7)$ and $(3,3)$

Now, $A=\dfrac{\partial^2 f}{\partial x^2}=6x$, $B=\dfrac{\partial^2 f}{\partial x\,\partial y}=12$ and $C=\dfrac{\partial^2 f}{\partial x\,\partial y}=6y$.

1. At (5, –1) , $A=30$, $B=12$, $C=-6$ and so $AC-B^2=-180-144=-324<0$.

Therefore, *f* has neither maximum nor minimum at (5, –1).

2. At (–1, 5), $A=-6$, $B=12$, $C=30$ and so $AC-B^2=-180-144=-324<0$.

Therefore, *f* has neither maximum nor minimum at (–1, 5).

3. At (–7, –7), $A=-42$, $B=12$, $C=-42$ and so $AC-B^2=1620>0$ and $A<0$.

Thus *f* has a maximum at $(-7,-7)$ and maximum value is *f*(-7,-7) = 784.

4. At $(3,3)$, $A=18$, $B=12$, $C=18$ and so $AC-B^2=324-144=180>0$ *and* $A=18>0$.

Thus *f* has a minimum at (3, 3) and minimum value is *f*(3,3) = -216.

1.8.28. Example. A rectangular box open from top is to have volume 3 cubic unit. Obtain the dimensions of the box requiring least material for its construction.

Solution. Let x, y, z be the edges of the open box and S be its surface. Since box is open, so

$$S = xy + 2yz + 2zx$$

Also it is given that $\quad x\,y\,z=32$

or $\quad z=\dfrac{32}{x\,y}$

Therefore, we have $\quad S=xy+2y\left(\dfrac{32}{x\,y}\right)+2x\left(\dfrac{32}{x\,y}\right) \;=x\,y+\dfrac{64}{x}+\dfrac{6\,y}{y}$

Then, $\dfrac{\partial S}{\partial x} = y-\dfrac{64}{x^2}$, $\dfrac{\partial S}{\partial y}=x-\dfrac{64}{y^2}$.

For extreme values, $\dfrac{\partial S}{\partial y}=0$ and $\dfrac{\partial S}{\partial y}=0$

and so $\quad y-\dfrac{64}{x^2}=0 \quad$ and $\quad x-\dfrac{64}{y^2}=0$.

Solving these, we get

$$x\left(64-x^3\right)=0$$

which implies, either $x=0$ or $64-x^3=0$,

that is, $x=0$ or $x=4$.

However, $x=0$ is not possible as in that case *y* does not exist.

When $x=4$ then $y=\dfrac{64}{16}=4$, and so stationary point is (4, 4).

Now, $A=\dfrac{\partial^2 S}{\partial x^2}=\dfrac{128}{x^3}$, $B=\dfrac{\partial^2 S}{\partial x \partial y}=1$, $C=\dfrac{\partial^2 S}{\partial y^2}=\dfrac{128}{y^3}$

Then, $AC-B^2=\dfrac{(128)^2}{x^3y^3}-1$.

Now, at (4, 4), $AC-B^2=\dfrac{(128)^2}{64\times 64}-1=4-1=3>0$.

Also, $A=\dfrac{128}{4^3}=2>0$

Hence S is least at (4, 4) and $z = \frac{32}{xy} = \frac{32}{4 \times 4} = 2$.

Therefore, dimensions of the box requiring least material for its construction are 4 ft and 2ft.

1.8.29. Lagrange's method of undetermined multipliers.

Lagrange's method of undetermined multipliers is used to find the extreme values of a function of three or more variables when the variables are not independent but have some relation between them.

Let $f(x, y, z)$ be the given function and the relation for x, y, z is

$$\phi(x, y, z) = 0 \qquad (1)$$

At a stationary point of $f(x, y, z)$,

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

Therefore,
$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad (2)$$

Differentiating (1), we get

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad (3)$$

Multiplying (3) by λ and adding to (2), we get

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial x}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right)dz = 0$$

Therefore,
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad (4)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad (5)$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad (6)$$

Solving (1), (4), (5) and (6), we obtain some values of x, y, z for which $f(x, y, z)$ is maximum or minimum.

1.8.30. Example. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$

Solution. Let $f(x, y, z) = x^2 + y^2 + z^2 \qquad (1)$

and $\phi(x, y, z) = x + y + z - 3a \qquad (2)$

Then, $f_x = 2x, f_y = 2y, f_z = 2z,$

and $\phi_x = \phi_y = \phi_z = 1,$

For stationary points, due to Lagrange's condition, we have

$$\frac{\partial f}{\partial x} + \lambda\,\frac{\partial \phi}{\partial x} = 0,\quad \frac{\partial f}{\partial y} + \lambda\,\frac{\partial \phi}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$

$$2x+\lambda=0 \qquad \text{or} \qquad x=\frac{-\lambda}{2}$$

$$2y+\lambda=0 \qquad \text{or} \qquad y=\frac{-\lambda}{2}$$

$$2z+\lambda=0 \qquad \text{or} \qquad z=\frac{-\lambda}{2}$$

Now $\quad x+y+z=3a \qquad (3)$

Using values of $x,\ y,\ z$ in (3), we get

$$\frac{-\lambda}{2}\ \frac{-\lambda}{2}\ \frac{-\lambda}{2}=3a$$

or $$\frac{-3\lambda}{2}=3a \qquad \text{or} \qquad \lambda=-2a$$

Therefore, $\quad x=a,\quad y=a,\quad z=a$

Differentiating (3) partially w.r.t. x and y, we have

$$1+0+\frac{\partial z}{\partial x}=0 \qquad \text{and} \qquad 0+1+\frac{\partial z}{\partial y}=0$$

$$\frac{\partial z}{\partial x}=-1 \qquad \text{and} \qquad \frac{\partial z}{\partial y}=-1$$

and from (1)

$$\frac{\partial f}{\partial x}=2x+0\ +\ 2z\frac{\partial z}{\partial x}=2x-\ 2z$$

$$\frac{\partial f}{\partial y}=0\ +\ 2y\ +\ 2z\ \frac{\partial z}{\partial y}\ =\ 2y\ -\ 2z$$

$$A=\frac{\partial^2 f}{\partial x^2}=2-2\frac{\partial z}{\partial x}=\ 2+2=4$$

$$C=\frac{\partial^2 f}{\partial y^2}=2-\frac{2\partial z}{\partial y}=2+2=4$$

$$B=\frac{\partial^2 f}{\partial x \partial y}\ =\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\ =\ \frac{\partial}{\partial x}\ (2y-2z)\quad =-2\frac{\partial z}{\partial x}=2$$

Thus, $\quad AC-B^2=(4)(4)-4=12\ >\ 0$ and $A=4\ >\ 0$.

Hence the given function has a minimum at the point given by (3) and minimum value is $x^2+y^2+z^2=a^2+a^2+a^2=3a^2$

1.8.31. Exercise.

1. Examine for maximum and minimum values of the following

(i) $x^3 - 3axy + y^3$ **(ii)** $y^2 + x^2 y + x^4$

(iii) $x^2 + y^2 + 6x + 12$ **(iv)** $x^3 + y^3 - 3x - 12y + 20$

(v) $x^2y^2 - 5x^2 - 8xy - 5y^2$ **(vi)** $xy(a - x - y)$

(vii) $f(x, y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$ $a, b > 0$

2. Show that $f(x, y) = (y - x)^4 + (x - 2)^4$ has minimum at (2, 2).

3. Show that $f(x, y) = (x - y)^2 + x^3 - y^3 + x^5$ has neither a maximum nor minimum at (0, 0).

4. Show

1. Verify Euler's theorem for $u = \frac{x^2 y^2}{x^2 + y^2}$.
2. Show that the surface area of a closed cuboid with square base and given volume is minimum when it is a cube.
3. Find the dimensions of a rectangular box without a top of maximum volume whose surface area is 108 sq.cms.

1.10. Summary. In this chapter, we discussed about various aspects of calculus, like differentiability, partial derivatives, total derivatives and their applications to the maximization and minimization problems.

Books Suggested.

1. Allen, B.G.D, Basic Mathematics, Mcmillan, New Delhi.
2. Volra, N. D., Quantitative Techniques in Management, Tata McGraw Hill, New Delhi.
3. Kapoor, V.K., Business Mathematics, Sultan chand and sons, Delhi.